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Invariant integration over the orthogonal group

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Abstract

I adapt a recently introduced method for integrating over the unitary group (Aubert S and Lam C S 2003 *J. Math. Phys.* **44** 6112–31) to the orthogonal group. I derive explicit formulae for a number of one, two and three-vector integrals, as well as recursion formulae for more complicated cases.

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1. Introduction

Integrals over compact Lie groups arise in physics naturally through applications of random matrix theory [1], which plays an important role in various fields of physics, ranging from nuclear theory [2], quantum chaos [3] and transport in mesoscopic devices [4], to quantum information theory [5]. A new application arose recently in the latter field from the study of statistics of interference in quantum algorithms [6]. Quantum algorithms can always be represented as real matrices by spending one more qubit (i.e. doubling the size of the Hilbert space), which labels the real or imaginary part of the wavefunction [7, 8], and the question was posed how this influences the amount of interference necessary in the quantum algorithm compared to a unitary representation [9]. Examining the statistics of interference drawn from the Haar orthogonal ensemble, i.e. the ensemble of orthogonal matrices with a flat distribution according to the Haar measure of the orthogonal group, then leads naturally to consider integrals of monomials of matrix elements over the orthogonal group $O(N)$.

Formulae for one-vector integrals were obtained earlier by Ullah [2] and by Mehta [10]. An n -vector integral is defined as an integral containing elements only from n columns (or from n rows) of the orthogonal matrix. Prosen and co-workers introduced an asymptotic method which works well for large N [11]. The problem was solved in principle by Gorin [12], who derived recursion relations connecting n -vector integrals to $(n - 1)$ -vector integrals. While the method is general, we will see below that explicit formulae can be obtained in a much easier way in various simple but important cases of integrals of low degree (and/or a small number of vectors). A number of interesting properties of integrals over the orthogonal (as well as the unitary and symplectic) group were also derived in [13], based on the use of Brauer algebras.

Recently, Aubert and Lam introduced a very elegant method for integration over the unitary group $U(N)$ solely based on the unitary constraint and the invariance of the Haar measure under unitary transformations [14]. This method is very powerful, and allowed the authors to obtain explicit formulae for the lowest order integrals, including all one-vector integrals, all two-vector integrals with up to three different matrix elements taken to arbitrary powers, and all two- and three-vector integrals up to order 6 (the order means here the number of matrix elements in the monomial to be integrated). Recursion relations can be obtained to reduce higher order integrals to these basic integrals. In the present paper, I adapt the method of invariant integration to the orthogonal group, and derive the corresponding basic integrals and recursion formulae.

2. Integrals over the orthogonal group

Consider orthogonal, real $N \times N$ matrices O with matrix elements O_{ij} , $1 \leq i, j \leq N$. We will be interested in integrals of the type

$$\mathcal{I}_{i_1 j_1 \dots i_p j_p} \equiv \int (\mathrm{d}O) O_{i_1 j_1} \cdots O_{i_p j_p}, \quad (1)$$

where the order p of the integral is some positive integer, and $(\mathrm{d}O)$ denotes the Haar invariant measure of the orthogonal group $O(N)$, normalized to $\int (\mathrm{d}O) = 1$. These integrals can be calculated based solely on

- the orthogonality relation $O^T O = \mathbf{1} = O O^T$ (where T denotes the transposed matrix), or explicitly

$$\sum_{i=1}^N O_{ij} O_{ik} = \delta_{jk} = \sum_{i=1}^N O_{ji} O_{ki}, \quad (2)$$

where δ_{jk} stands for the Kronecker-delta;

- and the invariance of the Haar measure, meaning that for any function $f(O)$ and an arbitrary (real) orthogonal ($N \times N$) matrix V ,

$$\int (\mathrm{d}O) f(O) = \int (\mathrm{d}O) f(V O) = \int (\mathrm{d}O) f(O V). \quad (3)$$

Here we will use for $f(O)$ the monomial

$$f(O) = \prod_{\lambda=1}^p O_{i_\lambda j_\lambda} \equiv O_{IJ}, \quad (4)$$

and we have introduced the notation O_{IJ} , where $I = \{i_1, \dots, i_p\}$, $J = \{j_1, \dots, j_p\}$. Correspondingly, $\mathcal{I}_{i_1 j_1 \dots i_p j_p}$ will be abbreviated as $\mathcal{I}_{i_1 j_1 \dots i_p j_p} \equiv \mathcal{I}(O_{IJ})$.

3. Relations from invariance

A number of powerful relations are obtained from choosing different explicit orthogonal matrices, under which $\mathcal{I}(O_{IJ})$ must be invariant.

3.1. Global sign change

The simplest example consists of $V = -\mathbf{1}$, i.e. $V_{ij} = -\delta_{ij}$, clearly an orthogonal transformation, with $O' = VO = -O$. Equation (3) implies $f(O) = (-1)^p f(O)$. Thus, p needs to be even, otherwise the integral will be zero. Even p will therefore be assumed from now on.

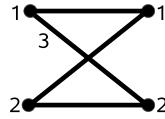


Figure 1. Example of a graphical representation for $\int(dO) O_{11} O_{12}^3 O_{21} O_{22}$. Each dot in the left column presents a left index of a matrix element, each dot in the right column a right index of a matrix element. Each line corresponds to a factor in the monomial, with its power written next to it. The power is equal to one if it is not written out explicitly. The number of lines entering each point must be even.

3.2. Local sign change

Another orthogonal transformation is induced by a matrix V with matrix elements $V_{ij} = (-1)^{s_i} \delta_{ij}$, where $s_i \in \{0, 1\}$. It leads to new matrix elements $O'_{ij} = (VO)_{ij} = (-1)^{s_i} O_{ij}$ and $f(O') = (-1)^{\sum_{\lambda=1}^p s_{i_\lambda}} f(O)$. As each s_{i_λ} is arbitrary $\in \{0, 1\}$, equation (3) implies that each i_λ has to appear an even number of times. The same reasoning applied to multiplication with V from the right lets us conclude that each j_λ has also to appear an even number of times, otherwise the integral will be zero.

3.3. Permutations

Consider the permutation that exchanges two indices, $i_0 \leftrightarrow j_0$. It is induced by a transformation V with matrix elements $V_{ij} = \delta_{ij}$ for $i \neq i_0, j_0$, $V_{i_0 j_0} = V_{j_0 i_0} = 1$, $V_{i_0 i_0} = V_{j_0 j_0} = 0$, which is manifestly orthogonal. An arbitrary permutation can be obtained by concatenating exchanges of two indices. The corresponding matrices are multiplied, and since the product of two orthogonal matrices is again orthogonal, an arbitrary permutation P can always be represented as real orthogonal matrix V . Multiplication from the left permutes the left-hand indices in O , $f(VO) = \prod_{\lambda} O_{P(i_\lambda)j_\lambda}$, and multiplication from the right permutes the right-hand indices, $f(OV) = \prod_{\lambda} O_{i_\lambda P(j_\lambda)}$. Therefore, the value of the indices is of no importance, the only thing which counts is the multiplicity of all different indices. Thus, we may rewrite I and J as $I = \{(i_1)^{\mu_1}, (i_2)^{\mu_2}, \dots, (i_t)^{\mu_t}\}$ and $J = \{(j_1)^{\nu_1}, (j_2)^{\nu_2}, \dots, (j_s)^{\nu_s}\}$, with $\sum_{i=1}^t \mu_i = p = \sum_{i=1}^s \nu_i$, where p is an even integer according to section 3.1, and all μ_i and ν_i must be even (section 3.2). We might even drop the indices i_λ and j_λ all together (e.g. chose them once for all as $i_\lambda = \lambda$ ($\lambda = 1, \dots, t$), $j_\lambda = \lambda$ ($\lambda = 1, \dots, s$)), and just keep the multiplicities $\mathcal{M} \equiv \{\mu_1, \dots, \mu_t\}$, $\mathcal{N} \equiv \{\nu_1, \dots, \nu_s\}$.

This result suggests a graphical representation of $\mathcal{I}(O_{IJ})$, where all first indices in O_{IJ} are represented by a dot in a left column (one dot for each different index), and all different second indices in O_{IJ} are represented by a dot in a right column. These dots are joined by lines, where each line represents a factor $O_{i_\lambda j_\lambda}$. If a factor appears to the power m_λ , we denote that power next to the line (see figure 1). If all powers m_λ are even, the integral is positive, and is called a ‘direct integral’; otherwise it is called ‘exchange integral’ [14]. In contrast to the integrals encountered for the unitary group [14], there are no complex conjugate factors here. This simplifies things at first, as we have to deal with only one type of lines, but also complicates things as it will lead to larger freedom in moving lines around (see below). Therefore, several recursion relations and their solutions will turn out to be more complicated than in the unitary case.

3.4. Transposition

We have $(dO) = (dO^T)$, and therefore $\mathcal{I}(O_{IJ}) = \mathcal{I}(O_{JI})$. In the graphical language this tells us that all diagrams are invariant under reflection at a central vertical line.

3.5. Rotations

Consider a rotation in the plane spanned by the basis vectors pertaining to the indices a and b , $V_{ij} = \delta_{ij}$ for $i \neq a, b$, $V_{aa} = V_{bb} = \cos \xi$, $V_{ab} = -V_{ba} = (\sin \xi)$. Multiplied from the right, this transforms matrix elements according to $O'_{ia} = (\cos \xi)O_{ia} - (\sin \xi)O_{ib}$, $O'_{ib} = (\sin \xi)O_{ia} + (\cos \xi)O_{ib}$, $O'_{ij} = O_{ij}$ otherwise. It is most instructive to consider the effect of this rotation for a simple example, the integral $\mathcal{I}((O_{11})^d)$, d even, and $a = 1, b = 2$. Expanding $(O'_{11})^d$, we obtain

$$\mathcal{I}((O'_{11})^d) = \sum_{e=0}^d \binom{d}{e} (\cos \xi)^{d-e} (-\sin \xi)^e \int (dO) (O_{11})^{d-e} (O_{12})^e = \mathcal{I}((O_{11})^d), \quad (5)$$

where the last equal sign is dictated by the invariance under rotations, so the left-hand side must be independent of ξ . As the only independent form of $\cos \xi$ and of $\sin \xi$ are powers of $\cos^2 \xi + \sin^2 \xi$, the left side can only contain even values of e , such that with $e = 2\tilde{e}$, $d = 2\tilde{d}$

$$\mathcal{I}((O'_{11})^d) = \sum_{\tilde{e}=0}^{\tilde{d}} \binom{2\tilde{d}}{2\tilde{e}} (\cos \xi)^{2(\tilde{d}-\tilde{e})} (\sin \xi)^{2\tilde{e}} \mathcal{I}((O_{11})^{2(\tilde{d}-\tilde{e})} (O_{12})^{2\tilde{e}}) \quad (6)$$

$$= \sum_{\tilde{e}=0}^{\tilde{d}} M_e (\cos \xi)^{2(\tilde{d}-\tilde{e})} (\sin \xi)^{2\tilde{e}}, \quad (7)$$

$$M_e = \binom{2\tilde{d}}{2\tilde{e}} \mathcal{I}((O_{11})^{2(\tilde{d}-\tilde{e})} (O_{12})^{2\tilde{e}}). \quad (8)$$

Invariance under rotations requires that $M_e \propto \binom{\tilde{d}}{\tilde{e}}$. The proportionality constant can be fixed from $\tilde{e} = 0$, $M_0 = \mathcal{I}((O_{11})^d) \equiv F_1(2\tilde{d})$, and we are thus led to $M_e = \binom{\tilde{d}}{\tilde{e}} \mathcal{I}((O_{11})^d)$. Together with (8) this leads to a series of relationships:

$$F_2(2(\tilde{d} - \tilde{e}), 2\tilde{e}) \equiv \mathcal{I}((O_{11})^{2(\tilde{d}-\tilde{e})} (O_{12})^{2\tilde{e}}) = \frac{\binom{\tilde{d}}{\tilde{e}}}{\binom{2\tilde{d}}{2\tilde{e}}} \mathcal{I}((O_{11})^d) = \frac{\binom{\tilde{d}}{\tilde{e}}}{\binom{2\tilde{d}}{2\tilde{d}}} F_1(2\tilde{d}). \quad (9)$$

In the graphical language, we have rotated $2\tilde{e}$ lines away from the right dot to a new, empty dot, and we have found a relationship between this new integral and the old one, containing only one line of multiplicity $d = 2\tilde{d}$. The same reasoning can be applied to a dot which is part of a more complicated diagram, i.e. a dot in which lines from several other dots terminate. The total number of lines arriving at the dot will again be denoted by d , and must still be even. Invariance under rotations implies again the same form (7) after expanding the powers of all matrix elements which get transformed, and leads to the same equation (9), where now $F_1(2\tilde{d})$ means the original diagram, and $F_2(2(\tilde{d} - \tilde{e}), 2\tilde{e})$ the diagram in which $2\tilde{e}$ lines have been rotated away from one dot to another, empty dot. Note that exactly the same relation is obtained for integration over the hypersphere (instead of integration over the orthogonal group) [14]. This is not surprising, as (5) is a one-vector integral, such that only the normalization of each column is involved, but no orthogonality relation between different columns. More generally, any one-vector integral over the orthogonal group $O(N)$ equals the corresponding integral over the hypersphere in N dimensions [12].

Equation (9) can be iterated to give the relations called ‘fan relations’ in [14],

$$F_t(2(\tilde{d} - \tilde{d}_1), 2(\tilde{d}_1 - \tilde{d}_2), \dots, 2(\tilde{d}_{t-2} - \tilde{d}_{t-1}), 2\tilde{d}_{t-1})) = F_1(2\tilde{d}) \frac{\binom{\tilde{d}}{\tilde{d}_1} \binom{\tilde{d}_1}{\tilde{d}_2} \cdots \binom{\tilde{d}_{t-2}}{\tilde{d}_{t-1}}}{\binom{2\tilde{d}}{2\tilde{d}_1} \binom{2\tilde{d}_1}{2\tilde{d}_2} \cdots \binom{2\tilde{d}_{t-2}}{2\tilde{d}_{t-1}}}. \quad (10)$$

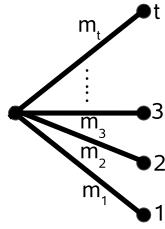


Figure 2. Graphical representation of the fan integral (equation (11)).

We recognize $2\tilde{d}_{t-1} = m_t$ as the multiplicity of the last line (line t), $2(\tilde{d}_{t-2} - \tilde{d}_{t-1}) = m_{t-1}$ as the multiplicity of line $t-1$, and so on, up to the multiplicity $m_1 = 2(\tilde{d} - \tilde{d}_1)$ of the first line. The relationships between \tilde{d}_i and m_j are easily inverted, and, when re-injected into (10), lead to the final form of the ‘fan relation’ for an integral represented by the diagram in figure 2,

$$F_t(m_1, \dots, m_t) = F_1(d) \frac{(d/2)!}{d!} \frac{m_1! \cdot \dots \cdot m_t!}{(\frac{m_1}{2})! \cdot \dots \cdot (\frac{m_t}{2})!}. \quad (11)$$

The integral for a single line with multiplicity d , $F_1(d)$, will be calculated in the next section.

4. Relations from orthogonality

The orthogonality relation (2) is a handy tool to reduce the number of lines in a diagram. It is most instructive to consider as simple example the integral

$$F_2(2(m-1), 2) = \int (dO) (O_{1k})^{2(m-1)} (O_{1l})^2 \quad k \neq l, \quad (12)$$

As we have seen, the value of the integral is independent of the index l , as long as $l \neq k$. At the same time, if we sum over *all* possible values of l (i.e. $l = 1, \dots, N$), we can make use of (2), as $\sum_{l=1}^N (O_{1l})^2 = 1$. Therefore,

$$\begin{aligned} (N-1)F_2(2(m-1), 2) &= \sum_{l=1}^N \int (dO) (O_{1k})^{2(m-1)} (O_{1l})^2 - \int (dO) (O_{1k})^{2m} \\ &= F_1(2(m-1)) - F_1(2m). \end{aligned} \quad (13)$$

This is an equation independent of the fan relation (11). The latter leads in the present case to

$$F_2(2(m-1), 2) = \frac{F_1(2m)}{2m-1}, \quad (14)$$

and combining this with (13) gives a recursion relation for $F_1(2m)$,

$$F_1(2m) = \frac{2m-1}{2m+N-2} F_1(2m-2), \quad (15)$$

with the obvious solution

$$F_1(2m) = \frac{(2m-1)!!(N-2)!!}{(2m+N-2)!!}. \quad (16)$$

The normalization $F_1(0) = \int (dO) = 1$ was used, and can be retrieved from (16) if we define $(-1)!! = 1$.

The orthogonality relation is useful if we have a dot in which only two lines end. Note that so far we have again used just the normalization of each line in the orthogonal matrix. If the two lines ending in dot l on the right were replaced by two lines originating from two different dots on the left, the sum over all values of l would give zero, regardless of the rest of the diagram.

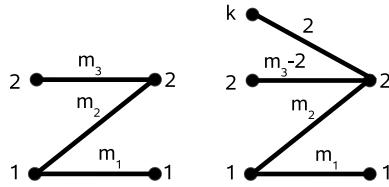


Figure 3. Graphical representation of the Z -integral, equation (17) (left) and the integral $\mathcal{I}(3b)$ used to establish a recursion relation for it (right).

5. Z -integrals

As an application of the techniques introduced, we now consider all possible integrals with up to three different factors, two indices on the right, and two on the left (see figure 3), baptised ‘ Z -integrals’ in [14],

$$Z(m_1, m_2, m_3) = \int (dO)(O_{11})^{m_1} (O_{12})^{m_2} (O_{22})^{m_3}. \quad (17)$$

All multiplicities m_1, m_2, m_3 must be even. We start from the integral $\mathcal{I}(3b)$ shown in figure 3, obtained by rotating two lines away from the first dot on the left to a new dot. The latter is arbitrary, and we can sum over it, avoiding the already taken indices 1 and 2, whereas in the full sum the extra line disappears,

$$\sum_{k=3}^N \mathcal{I}(3b) = (N-2)\mathcal{I}(3b) = Z(m_1, m_2, m_3 - 2) - Z(m_1, m_2, m_3) - Z(m_1, m_2 + 2, m_3 - 2). \quad (18)$$

On the other hand, the fan relation (11) can be applied to the upper two lines of the diagram $(3b)$, which gives $\mathcal{I}(3b) = \frac{1}{m_3-1} Z(m_1, m_2, m_3)$. Once we insert the latter result into (18), we find the recursion

$$Z(m_1, m_2, m_3) = \frac{m_3 - 1}{N + m_3 - 3} (Z(m_1, m_2, m_3 - 2) - Z(m_1, m_2 + 2, m_3 - 2)), \quad (19)$$

which should be supplemented by the initial value

$$Z(m_1, m_2, 0) = F_2(m_1, m_2) = \frac{(m_1 + m_2 - 1)!!(N - 2)!!}{(m_1 + m_2 + N - 2)!!} \frac{\left(\frac{m_1+m_2}{2}\right)! m_1! m_2!}{(m_1 + m_2)!\left(\frac{m_1}{2}\right)!\left(\frac{m_2}{2}\right)!} \quad (20)$$

obtained from the fan relation (11). If we apply the recursion relation again, we easily convince ourselves that its solution is of the form

$$Z(m_1, m_2, m_3) = \frac{(m_3 - 1)!!(N - 3)!!}{(N + m_3 - 3)!!} \sum_{i=0}^{m_3/2} \binom{m_3/2}{i} Z(m_1, m_2 + 2i, 0) (-1)^i. \quad (21)$$

A closed form can be found for the sum in this equation, if we consider N even and odd separately. For N even, (20) can be re-expressed as

$$Z(m_1, m_2, 0) = \frac{\left(\frac{N-2}{2}\right)! m_1! m_2!}{2^{m_1+m_2} \left(\frac{m_1}{2}\right)!\left(\frac{m_2}{2}\right)!\left(\frac{m_1+m_2+N-2}{2}\right)!}, \quad (22)$$

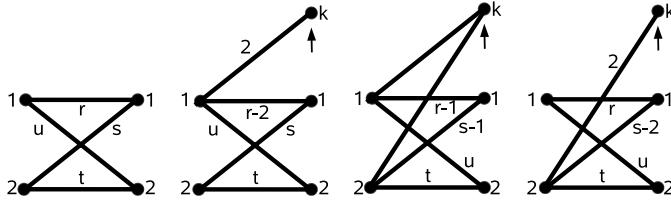


Figure 4. The exchange integral $X(r, s, t, u)$ (left) and the integrals $\mathcal{I}(4a)$, $\mathcal{I}(4b)$, and $\mathcal{I}(4c)$ used to calculate it (second from left to right, respectively). The latter integrals are obtained by rotating two lines away from dot 1 on the right to a new dot $k \neq 1, 2$; the index k is then summed over.

such that

$$\begin{aligned} Z(m_1, m_2, m_3) &= \frac{(m_3 - 1)!!(N - 3)!!}{(N + m_3 - 3)!!} \frac{\left(\frac{N-2}{2}\right)! m_1!}{2^{m_1+m_2} \left(\frac{m_1}{2}\right)!} S_1, \\ S_1 &= \sum_{i=0}^{m_3/2} \left(\frac{-1}{4}\right)^i \binom{m_3/2}{i} \frac{(m_2 + 2i)!}{\left(\frac{m_2}{2} + i\right)! \left(\frac{m_1+m_2+2i+N-2}{2}\right)!} \\ &= \frac{2^{\frac{m_2-m_3}{2}} (m_2 - 1)!! (m_1 + m_3 + N - 3)!!}{(m_1 + N - 3)!! \left(\frac{m_1+m_2+m_3+N-2}{2}\right)!}. \end{aligned} \quad (23)$$

This leads to the final result

$$\begin{aligned} Z(m_1, m_2, m_3) &= \frac{m_1! m_2! m_3! \left(\frac{m_1+N-2}{2}\right)! \left(\frac{m_3+N-2}{2}\right)!}{2^{m_1+m_2+m_3} (m_1 + N - 2)! (m_3 + N - 2)! \left(\frac{m_1+m_3+N-2}{2}\right)! \left(\frac{m_1+m_2+m_3+N-2}{2}\right)!} \\ &\times \frac{(m_1 + m_3 + N - 2)! (N - 2)!}{\left(\frac{m_1}{2}\right)! \left(\frac{m_2}{2}\right)! \left(\frac{m_3}{2}\right)!} \end{aligned} \quad (24)$$

$$= \frac{2^{2-N} \Gamma\left(\frac{1+m_1}{2}\right) \Gamma\left(\frac{1+m_2}{2}\right) \Gamma\left(\frac{1+m_3}{2}\right) \Gamma(N - 1) \Gamma\left(\frac{1}{2}(N + m_1 + m_3 - 1)\right)}{\pi \Gamma\left(\frac{1}{2}(N + m_1 - 1)\right) \Gamma\left(\frac{1}{2}(N + m_3 - 1)\right) \Gamma\left(\frac{1}{2}(N + m_1 + m_2 + m_3)\right)}. \quad (25)$$

The latter form turns out to be valid also for odd N .

6. Exchange integrals

Exchange integrals contain at least one line with odd multiplicity, and can therefore be positive or negative. The total number of lines arriving in any dot must, of course, still be even, otherwise the integral vanishes. Consider the structurally simplest exchange integral depicted in figure 4,

$$X(r, s, t, u) \equiv \int (dO) (O_{11})^r (O_{21})^s (O_{22})^t (O_{12})^u, \quad (26)$$

$r, s, t, u \in \mathbb{N}$. The case $r = s = t = u = 1$ is easily solved by summing over the index of an arbitrary point, say the upper right one. We obtain

$$X(1, 1, 1, 1) = -\frac{1}{N - 1} F_2(2, 2) = -\frac{1}{(N - 1)N(N + 2)}. \quad (27)$$

For general r, s, t, u we derive a recursion relation using invariance under a rotation between indices 1 and 3 (where 3 is a new index), multiplied from the right. This gives

$$\begin{aligned} X(r, s, t, u) &= X'(r, s, t, u) \equiv \int (\mathrm{d}O) ((\cos \xi) O_{11} - (\sin \xi) O_{13})^r \\ &\quad \times ((\cos \xi) O_{21} - (\sin \xi) O_{23})^s O_{22}^t O_{12}^u \\ &= (\cos \xi)^{r+s} M_0 + (\cos \xi)^{r+s-2} (\sin \xi)^2 M_1 + \dots \end{aligned} \quad (28)$$

where $M_0 = X(r, s, t, u)$ is evident from choosing $\xi = 0$, and M_1 is given by the diagrams in figure 4 with corresponding values $\mathcal{I}(4a)$, $\mathcal{I}(4b)$ and $\mathcal{I}(4c)$,

$$M_1 = \frac{r(r-1)}{2} \mathcal{I}(4a) + rs \mathcal{I}(4b) + \frac{s(s-1)}{2} \mathcal{I}(4c). \quad (29)$$

According to section 3.5, we must have $M_1 = \frac{r+2}{2} X(r, s, t, u)$. The order of the integrals $\mathcal{I}(4a)$, $\mathcal{I}(4b)$ and $\mathcal{I}(4c)$ can again be reduced by summing over the index of the new point, which yields

$$\mathcal{I}(4a) = \frac{1}{N-2} (X(r-2, s, t, u) - X(r, s, t, u) - X(r-2, s+2, t, u)) \quad (30)$$

$$\mathcal{I}(4b) = -\frac{1}{N-2} (X(r, s, t, u) + X(r-1, s-1, t+1, u+1)) \quad (31)$$

$$\mathcal{I}(4c) = \frac{1}{N-2} (X(r, s-2, t, u) - X(r, s-2, t+2, u) - X(r, s, t, u)). \quad (32)$$

If we insert these integrals into (29), we get the recursion relation

$$\begin{aligned} X(r, s, t, u) &= \frac{1}{(r+s)(N-2)+r(r-1)+2rs+s(s-1)} (-2rs X(r-1, s-1, t+1, u+1) \\ &\quad + r(r-1)(X(r-2, s, t, u) - X(r-2, s+2, t, u)) \\ &\quad + s(s-1)(X(r, s-2, t, u) - X(r, s-2, t+2, u))). \end{aligned} \quad (33)$$

No closed solution of this recursion relation could be found, but the principle of its application is clear: iterating (33) will reduce the multiplicities of the lines with power r or s , till a power zero is achieved, and then the integral is reduced to a Z-integral, whose value is known for all multiplicities of the remaining lines; see equation (23). Note that the formula corresponding to (33) for the unitary case is quite different, as one must rotate always one line corresponding to a non-conjugated matrix element and one line corresponding to a complex-conjugated matrix element [14].

7. Integrals of order 6

In this final section, I will give explicit formulae for all integrals of order $p = 6$, which are not of the fan-type or Z-type. They are shown in figure 5, and will be denoted accordingly $\mathcal{I}(5a)$, $\mathcal{I}(5b)$, $\mathcal{I}(5c)$, $\mathcal{I}(5d)$, $\mathcal{I}(5e)$, $\mathcal{I}(5f)$, $\mathcal{I}(5g)$. All of them are obtained by summing over the index indicated by an arrow. We are led to

$$\mathcal{I}(5a) = \frac{1}{N-1} (F_2(2, 2) - F_3(2, 2, 2)) = \frac{N+3}{(N-1)N(N+2)(N+4)} \quad (34)$$

$$\mathcal{I}(5b) = \frac{1}{N-2} (Z(2, 0, 2) - 2\mathcal{I}(5a)) = \frac{-2+N(N+3)}{(N-2)(N-1)N(N+2)(N+4)} \quad (35)$$

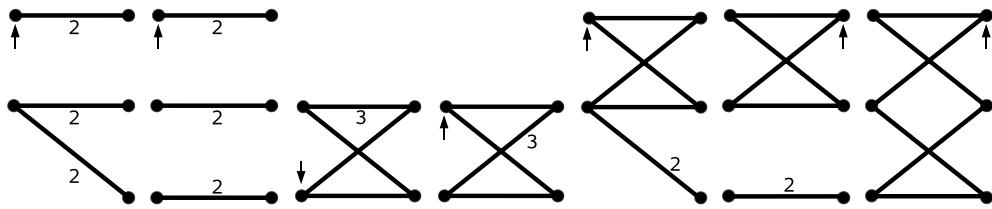


Figure 5. All the integrals of degree $p = 6$ which are not of the fan-type or Z-type considered previously. These integrals are labeled $\mathcal{I}(5a), \mathcal{I}(5b), \mathcal{I}(5c), \mathcal{I}(5d), \mathcal{I}(5e), \mathcal{I}(5f), \mathcal{I}(5g)$ (from left to right), respectively, and can be obtained by summing over the indices of the dots with an arrow. $\mathcal{I}(5c)$ and $\mathcal{I}(5d)$ are of the exchange type considered in figure 4.

$$\mathcal{I}(5c) = -\frac{1}{N-1} F_2(2, 4) = -\frac{3(N-2)!!}{(N-1)(N+4)!!} = \mathcal{I}(5d) \quad (36)$$

$$\mathcal{I}(5e) = -\frac{1}{N-1} F_3(2, 2, 2) = -\frac{(N-2)!!}{(N-1)(N+4)!!} \quad (37)$$

$$\mathcal{I}(5f) = -\frac{1}{N-2} (\mathcal{I}(5a) + \mathcal{I}(5e)) = -\frac{1}{(N-2)(N-1)N(N+4)} \quad (38)$$

$$\mathcal{I}(5g) = -\frac{2}{N-2} \mathcal{I}(5e) = \frac{2(N-2)!!}{(N-2)(N-1)(N+4)!!}. \quad (39)$$

8. Summary

As a summary, I have adapted the method of invariant integration introduced in [14] to the case of integration over the orthogonal group $O(N)$. Explicit formulae were obtained for all one-vector integrals (these coincide with integrals over a hypersphere in N dimensions), for all two-vector integrals with up to three different matrix elements, as well as for all integrals up to order 6. In more complicated cases recursion relations were derived, in particular for all exchange integrals with two different indices on the left and two on the right.

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References

- [1] Guhr T, Müller-Gröling A and Weidenmüller H A 1998 *Phys. Rep.* **299** 190
- [2] Ullah N 1964 *Nucl. Phys.* **58** 65
- [3] Haake F 1991 *Quantum Signatures of Chaos* (Berlin: Springer)
- [4] Baranger H U and Mello P A 1994 *Phys. Rev. Lett.* **73** 142
- [5] Życzkowski K and Sommers H-J 2003 *J. Phys. A: Math. Gen.* **36** 10115
- [6] Braun D and Georgeot B 2006 *Phys. Rev. A* **73** 022314
- [7] Shi Y 2002 *Preprint* quant-ph/0205115
- [8] Aharonov D 2003 *Preprint* quant-ph/0301040

- [9] Arnaud L and Braun D *Preprint* at press
- [10] Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
- [11] Prosen T, Seligman T H and Weidenmueller H 2002 *Preprint* [math-ph/0203042](#)
- [12] Gorin T 2002 *J. Math. Phys.* **43** 3342
- [13] Collins B and Śniady T 2006 *Commun. Math. Phys.* **264** 773–95 (*Preprint* [math-ph/0402073](#))
- [14] Aubert S and Lam C 2003 *J. Math. Phys.* **44** 6112